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Zeroth Law of Thermodynamics and Transitivity of Simultaneity

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The transitivity of thermal equilibrium is equivalent to the transitivity of clock rate synchronization. The condition of clock rate synchronization in general relativity is shown, which is weaker than time-orthogonality.

1. INTRODUCTION

There are two important laws of transitivity in physics (Zhao, 1991). One is the zeroth law of thermodynamics, which states the transitivity of thermal equilibrium. The other is the law of the transitivity of simultaneity in general relativity (Landau and Lifshitz, 1975), which says that one can synchronize coordinate clocks placed along a closed path to a simultaneous moment if and only if the reference frame is time-orthogonal.

In this paper, we will give a fundamental and intrinsic relationship between the two laws on transitivity in physics. The transitivity of thermal equilibrium is equivalent to the transitivity of clock rate synchronization.

In Section 2 we show the relationship between the transitivity of thermal equilibrium and the transitivity of clock rate synchronization by means of the thermal Green function. In Section 3 we discuss "clock synchronization" and "clock rate synchronization" in general relativity, and give a condition on "clock rate synchronization" which is different from that given by Landau and Lifshitz. We present a conclusion and discussion in Section 4.

2. THERMAL GREEN FUNCTION AND THERMAL EQUILIBRIUM TRANSITIVITY

The question is how to establish a link between time and temperature. One approach is through the thermal Green function in quantum field theory,

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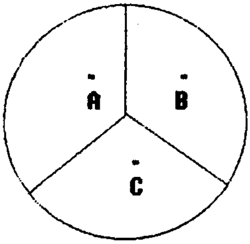

Fig. 1. Three systems with adiabatic partitions. Each has a standard clock to measure its proper time.

where temperature appears as a pure imaginary time period in the thermal green function (Gibbons and Perry, 1978).

Let us consider the simple case of the spin-0 Bose–Einstein ideal gas. Three adjacent macroscopically infinitesimal systems A, B, and C are located in a 4-dimensional Riemann spacetime (see Fig. 1). Each system has a standard clock. Adiabatic partitions are used to separate the three systems. When the three systems are in their respective thermal equilibrium, their thermal Green's functions are given as (Hartle and Hawking, 1976; Gibbons and Perry, 1978)

$$
\begin{aligned}
G_a(\Delta\tau_a) &= G_a(\Delta\tau_a + i\beta_{pa}) \\
G_b(\Delta\tau_b) &= G_b(\Delta\tau_b + i\beta_{pb}) \\
G_c(\Delta\tau_c) &= G_c(\Delta\tau_c + i\beta_{pc})
\end{aligned}
\tag{1}
$$

where τ_a, τ_b, and τ_c are proper times indicated by the standard clocks A, B, and C, respectively, and

$$T_p = 1/\beta_p \tag{2}$$

is the proper temperature, and β_p is the imaginary time period corresponding to proper time τ.

However, proper variables are local quantities that are applicable only to local measurements. We need global coordinate quantities to represent physical laws in the whole curved spacetime. It is known that the proper quantities are related to their corresponding coordinate quantities by a redshift factor. Hence, the above thermal Green functions can be expressed by coordinate quantities

$$
\begin{aligned}
G_a(\Delta t_a) &= G_a(\Delta t_a + i\beta_a) \\
G_b(\Delta t_b) &= G_b(\Delta t_b + i\beta_b) \\
G_c(\Delta t_c) &= G_c(\Delta t_c + i\beta_c)
\end{aligned}
\tag{3}
$$

We have

$$\Delta t = \Delta\tau/\sqrt{-g_{00}} \tag{4}$$

$$T = T_p \cdot \sqrt{-g_{00}} \tag{5}$$

$$\beta = \beta_p/\sqrt{-g_{00}} \tag{6}$$

where t is coordinate time, T is coordinate temperature, and β is the imaginary time period corresponding to coordinate time t (Einstein, 1955; Birrell and Davis, 1982).

Consider the transitive property of thermal equilibrium among the systems A, B, and C. The thermal equilibrium between two systems in Riemann spacetime means that they have the same coordinate temperature, while their proper temperatures are different in most cases. If the system A is in thermal equilibrium with B, we have

$$\beta_a = (1/T_a)\delta_a = \beta_b = (1/T_b)\delta_b \tag{7}$$

We note that β in the thermal Green function depends on two factors. T is a thermal characteristic of the equilibrium state; in fact, T is the temperature magnitude. δ is the unit scale of imaginary time. Because $i^2 = -1$, the absolute value of the unit of imaginary time is equal to that of real time. So δ is also the rate of the coordinate clock.

In the nonrelativistic case, we can assume that the clock rates are the same everywhere, or the unit magnitude of time is the same in whole space. In special relativity, we can always synchronize clocks placed at different space points. Their clock rates (the unit magnitude of time) can be adjusted to the same rate by means of the invariance of the light velocity and spacetime homogeneity and isotropy. In these two cases, β in the thermal Green function only depends on the thermal characteristic of the equilibrium state. And one must get $\beta_a = \beta_b$ if system A is in thermal equilibrium with B.

The problem of clock rate synchronization is more complicated in a curved spacetime. Generally speaking, the rates of standard clocks at rest at different space points are different. Further, the rates of coordinate clocks can be synchronized only in some cases. This means that we cannot identify the unit magnitude of coordinate time at different space points in most cases.

Hence, equation (7) may not be valid if we only know that system A is in thermal equilibrium with B, but we do not know if the rates of clocks are synchronized or not.

It is known that the "coordinate temperature" is an indicator for a thermal equilibrium state in curved spacetime. If thermal equilibrium is transitive in spacetime, then the coordinate temperature must be the same for all the parts in the system, or equation (7) must be valid. So we are sure that rate of clock

A has not been synchronized with clock B if equation (7) is still not valid although system A is in thermal equilibrium with B.

Now we discuss the relationship between the transitive property of thermal equilibrium and the transitive property of "clock rate synchronization." First, we assume that system A is in thermal equilibrium with B when we take away the adiabatic partition between A and B (but retain the rigid diathermic partition). The thermal Green functions for systems A and B are

$$\begin{aligned} G_a(\Delta t_a) &= G_a(\Delta t_a + i\beta_a) \\ G_b(\Delta t_b) &= G_b(\Delta t_b + i\beta_b) \end{aligned} \tag{8}$$

where t_a and t_b are coordinate times, and β_a and β_b are imaginary time periods corresponding to t_a and t_b respectively. If $\beta_a \neq \beta_b$, we can change the rate of coordinate clock B to make sure that

$$\beta_b = \beta_a \tag{9}$$

Therefore, the rate of clock B is the same as A at present. It follows that their unit times δ_a and δ_b satisfy

$$\delta_b = \delta_a \tag{10}$$

Second, we assume that system B is also in thermal equilibrium with C and their thermal Green functions do not change after taking away the adiabatic partition between them (but retain the rigid diathermic partition)

$$\begin{aligned} G_b(\Delta t_b) &= G_b(\Delta t_b + i\beta_b) \\ G_c(\Delta t_c) &= G_c(\Delta t_c + i\beta_c) \end{aligned} \tag{11}$$

Similarly, we can change the rate of coordinate clock C to achieve

$$\beta_c = \beta_b \tag{12}$$

Therefore, the rates of the two clocks are the same

$$\delta_c = \delta_b \tag{13}$$

If thermal equilibrium is transitive, system A should be in thermal equilibrium with C. β_{pc} and β_{pa} should not vary after taking away the adiabatic partition between systems A and C (retaining the rigid diathermic partition). G_a and G_c do not change, nor do β_a and β_c. From equations (9) and (12), we know that

$$\beta_a = \beta_c \tag{14}$$

so

$$\delta_a = \delta_c' \tag{15}$$

It should be noted that δ_c' is obtained by synchronizing the rates of coordinate clocks C and A (i.e., from $\beta_a = \beta_c$). So δ_c' is different from δ_c in equation (13), where δ_c is obtained by synchronizing the rates of coordinate clocks C and B (i.e., from $\beta_c = \beta_b$). From equations (10) and (13), we know that

$$\delta_a = \delta_b = \delta_c \tag{16}$$

so

$$\delta_c = \delta_c' \tag{17}$$

We come to the conclusion that the transitive property of thermal equilibrium leads to the transitive property of synchronization of rates of coordinate clocks.

If thermal equilibrium is not transitive (i.e., the zeroth law is violated), system A is not in thermal equilibrium with C, although system A is in thermal equilibrium with B, and B is also in thermal equilibrium with C. We insert adiabatic partitions back between systems A and B, as well as between B and C, then take away the adiabatic partition between systems A and C (retaining the rigid diathermic partitions). Now, systems A and C will relax to a new thermal equilibrium state. Their thermal Green functions will change. It is convenient to assume that the heat capacity of system A is much larger than that of C. So, β_c will change to β_c'', but the change of β_a can be neglected. We have

$$\begin{aligned} G_c(\Delta t_c) &= G_c(\Delta t_c + i\beta_c'') \\ G_a(\Delta t_a) &= G_a(\Delta t_a + i\beta_a) \end{aligned} \tag{18}$$

Evidently,

$$\beta_c'' \neq \beta_b = \beta_a \tag{19}$$

In order to change β_c'' to

$$\tilde{\beta}_c = \beta_a \tag{20}$$

we have to adjust the rate of the coordinate clock C to enable β_c'' to change to $\tilde{\beta}_c$. The new rate of the clock C is

$$\tilde{\delta}_c \neq \delta_c \tag{21}$$

It should be noticed that the new rate $\tilde{\delta}_c$ comes from synchronization with clock A, but the old rate δ_c comes from synchronization with clock B. Equation (21) tells us that synchronization of rates of coordinate clocks is not transitive when thermal equilibrium is not transitive.

Thus, the transitive property of thermal equilibrium is a necessary and sufficient condition for the transitive property of the synchronization of rates of coordinate clocks.

3. CLOCK SYNCHRONIZATION IN GENERAL RELATIVITY

Landau and Lifshitz (1975) indicate that the simultaneity of the space point A with B in a curved spacetime can be described by the time difference of their coordinate clocks:

$$\Delta t = t_a - t_b = -(g_{0i}/g_{00})\, dx^i \qquad (i = 1, 2, 3) \tag{22}$$

In general, Δt is not an exact differential

$$\oint \Delta t \neq 0 \tag{23}$$

Therefore, one can synchronize coordinate clocks placed along a closed path at a simultaneous moment if we have

$$g_{0i} = 0 \tag{24}$$

or

$$\oint \Delta t = 0 \tag{25}$$

So Landau and Lifshitz show that simultaneity is transitive if and only if a spacetime is time-orthogonal. In other words, one will not have simultaneity surfaces unless the coordinate system is time-orthogonal.

However, condition (24) is too strong for our question. We do not need to synchronize simultaneous moments of coordinate clocks. It is enough to have the "synchronization of clock rate" for the study of the relationship between time and thermal equilibrium. Now, let us give the condition on the transitivity of clock rate synchronization.

At the first simultaneous moment of the space points A and B, the time difference of their coordinate clocks is

$$\Delta t_1 = t_{a1} - t_{b1} = -(g_{0i}/g_{00})_1\, dx^i \tag{26}$$

At the second simultaneous moment, the time difference is

$$\Delta t_2 = t_{a2} - t_{b2} = -(g_{0i}/g_{00})_2\, dx^i \tag{27}$$

The difference between the rates of the two clocks can be obtained as

$$\begin{aligned}\delta(\Delta t) &= (\Delta t_a) - (\Delta t_b) = (t_{a2} - t_{a1}) - (t_{b2} - t_{b1}) \\ &= (t_{a2} - t_{b2}) - (t_{a1} - t_{b1}) \\ &= -[(g_{0i}/g_{00})_2 - (g_{0i}/g_{00})_1]\, dx^i\end{aligned} \tag{28}$$

Therefore, the rates of the coordinate clocks are the same everywhere, or the synchronization of clock rates is transitive, if and only if (g_{0i}/g_{00}) is independent of the coordinate time t,

$$\partial(g_{0i}/g_{00})/\partial t = 0 \tag{29}$$

or

$$\oint \delta(\Delta t) = 0 \tag{30}$$

This condition is weaker than the time-orthogonality. Obviously, equation (29) is only a necessary condition, but not a sufficient condition for constructing simultaneity surfaces.

4. CONCLUSION AND DISCUSSION

There exists a fundamental and intrinsic relationship between thermal equilibrium transitivity and clock rate synchronization. The condition of clock rate synchronization in general relativity as given by us is weaker than the time-orthogonality necessary to establish a simultaneity surface.

We find that the time properties in general relativity are closely related to the laws of thermodynamics.

The zeroth law of thermal equilibrium transitivity is equivalent to the clock rate synchronization in curved spacetime, as can be shown by means of the thermal Green function. The first law of thermodynamics, i.e., the law of conservation of energy, implies time homogeneity. The second law of thermodynamics indicates a time arrow for a large system. In another paper, we will point out that the problem of singularity in general relativity may be interpreted by the third law of thermodynamics. Time in general relativity has neither a "beginning" nor an "end" according to the third law of thermodynamics.

REFERENCES

Birrell, N. D., and Davis, P. C. W. (1982). *Quantum Fields in Curved Space*, Cambridge University Press, Cambridge, p. 27.

Einstein, A. (1955) *The Meaning of Relativity*, 5th ed., Princeton University Press, Princeton, New Jersey.

Gibbons, G. W., and Perry, M. J. (1978) *Proceedings of the Royal Society of London A*, **358**, 467–494.

Hartle, J. B., and Hawking, S. W. (1976). *Physical Review D*, **13**, 2188.

Landau, L. D., and Lifshitz, E. M. (1975). *The Classical Theory of Field*, Pergamon Press, 234.

Zhao Zheng (1991). *Science in China A*, **34** (7), 835–840.